Generating Multistep Methods for Special Ordinary Differential Equations of Higher-Order

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It is possible to integrate a differential equation of higher-order of the form $y^{(n)} = f(x, y, y', y'', ..., y^{(n-1)})$ by reducing it to the system $y' = y_1$, $y_1' = y_2$, ..., $y_{n-2}' = y_{n-1}$, $y_{n-1}' = f(x, y, y_1, y_2, ..., y_{n-1})$, and applying one consistent and stable method for systems of equations of the first order. This procedure is a perfectly legitimate one. No accuracy is lost, nor is there any unnecessary outlay of computional effort.

The situation is slightly different if the equation to be integrated is of the form $y^{(n)} = f(x, y)$, $n > 2$, where no derivates appear in the right-hand side member of the differential equation. Equations of this type are called *special* differential equations. If one is not particularly interested in the values of the intermediate derivates, it seems unnatural to introduce them artificially in order to produce systems of first order equations.

Therefore, it is interesting to have consistent and stable methods to solve directly these equations. In this paper, it is proposed a general framework for linear multistep methods for special ordinary differential equations of higher-order of the form $\sum_{i=0}^{k} \alpha_i y_{p+i} = h^n \sum_{i=0}^{k} \beta_i f_{p+i}$. In this method framework, $y_p, ..., y_{p+k}$ are the last approximations of the solutions of the differential equation; $f_p, ..., f_{p+k}$ are the last n-derivates of the differential equation, and h is the step used.

This framework is such that a method for an equation of order n has as its α coefficients the coefficients of the $(1-x)^n$ Newton's Binomial: $\alpha_i = (-1)^{n-i} C_{n,i}$, and $\sum_{i=0}^{k} \beta_i = 1$, where $C_{n,i}$ is the combination of n taken i [6].

It is obvious that $n = k$ for this framework. This family of methods can be proved consistent and stable, and, hence, convergent. This is done using a generalized definition of stability and convergence for a multistep method for special ordinary differential equations of higher-order, which are similar to the definition of stability for a multistep method for special ordinary differential equations of the first-order and of the second-order.

The definition of consistence for a multistep method for special ordinary differential equations of higher-order is similar to the definition of consistence for a multistep method for special ordinary differential equations of the first-order and of the second-order [2, 4, 5]. To define the consistence of this kind of methods, the definition of the difference operator presented in [2, 4, 5] is extended for special ordinary differential equations of higher-order: $L[y(x); h] = \sum_{q=0}^{\infty} C_q h^q y^{(q)}(x)$, where

$$\begin{array}{lll} C_q = \frac{1}{q!} \sum_{i=0}^{k} i^q \alpha_i, & q < n & \text{or} \\ C_q = \frac{1}{q!} \sum_{i=0}^{k} i^q \alpha_i - \frac{1}{(q-n)!} \sum_{i=0}^{k} i^{q-n} \beta_i, & q \geq n & \end{array}$$

where h is the step of the method and n is the order of the special ordinary differential equation.

The consistence is defined by the following definition: a multistep method for special ordinary differential equations of order n is consistent if and only if $\forall i (i \leq n \rightarrow C_i = 0)$.

Finally, the order of consistence of a multistep method for special ordinary differential equations of order n is p where $\forall i (i < p + n \rightarrow C_i = 0) \wedge C_{p+n} \neq 0$. That is, the minimum order of consistence of a consistent multistep method for special ordinary differential equations is 1.

To prove the consistence of the family of methods presented in this text, one can use induction on n and the fact that $C_{n,i} = C_{n-1,i} + C_{n-1,i-1}$, and the technics shown in [6]. This proves that

$$\forall q < n \rightarrow C_q = \frac{1}{q!} \sum_{i=0}^{n} i^q (-1)^{n-i} C_{n,i} = 0$$

$$C_n = \frac{1}{n!} \sum_{i=0}^{n} i^n (-1)^{n-i} C_{n,i} - \frac{1}{(n-n)!} \sum_{i=0}^{n} i^{n-n} \beta_i = 0$$

Therefore, since $C_0 = C_1 = ... = C_n = 0$, the framework of methods is consistent.

The definition of stability for a multistep method for special ordinary differential equations of higher-order is similar to the definition of stability for a multistep method for special ordinary differential equations of the first-order and of the second-order [2, 4, 5]. The definition of stability presented in [2, 4, 5], which is also extended for special ordinary differential equations of higher-order, is: a multistep method for special ordinary differential equations of order n is stable if and only if the modulus of no root of the polynomial associated to the α coefficients, $\rho(x) = \alpha_k x^k + \alpha_{k-1} x^{k-1} + ... + \alpha_0$ exceed 1 and the multiplicity of the roots of modulus 1 must be at most n, where n is the order of the special ordinary differential equation.

In the family of methods used, where $\alpha_i = (-1)^{n-i} C_{n,i}$, the polynomial associated to the α coefficients is $\rho(x) = C_{n,n} x^n - C_{n,n-1} x^{n-1} + ... + (-1)^n C_{n,0} = (x-1)^n$. All the n roots of this polynomial equals 1. Therefore, this family of methods is stable.

This framework has the values of the α coefficients fixed, but the same cannot be said about the values of the β coefficients. The only imposition on the β coefficients is that their sum must equal 1. Consequently, their values can be determined in order to increase the consistence order. This is done using the consistence definition, which leads to a linear system of equations that, when solved, gives the desired β coefficients. That is, the aim is to find β coefficients in order to make some $C_q = 0$, $q > n$. To do this, since there are $n+1$ β coefficients, a system of $n+1$ equations can be used to find the values of the β coefficients. Once the first equation is determined by $C_n = 0$ (which says that the sum of the β coefficients must equal 1), the equations will range from $C_n = 0$ to $C_{2n} = 0$. Therefore:

$$\begin{array}{rlcl}
C_n = 0: & \sum_{i=0}^{n} \beta_i & = & 1 \\
C_{n+1} = 0: & (n+1) \sum_{i=0}^{n} i\beta_i & = & \sum_{i=0}^{n} i^{n+1} (-1)^{n-i} C_{n,i} \\
 & ... & & \\
C_{2n} = 0: & 2n \times (2n-1) \times ... \times (n+1) \sum_{i=0}^{n} i^n \beta_i & = & \sum_{i=0}^{n} i^{2n} (-1)^{n-i} C_{n,i}
\end{array}$$

Since $C_0 = C_1 = ... = C_{2n} = 0$, the consistence order of the method is at least n.

As future work, it is intended to study if there is another consistent and stable family of methods whose α coefficients are given by a different Newton's Binomial. If so, it is interesting to investigate if its consistence order can be greater than the one of the family of methods shown here for the same order of the special ordinary differential equation. Furthermore, it is intended to implement a generator of methods using the framework described in this paper which produces a method with the greatest consistence order possible as described in this work.

References

[1] Boyce, W. E., Prima, R. C. Di, Elementary Differential Equations and Boundary Value Problems, 5^{th} Edition, John Wiley & Sons (1992).

[2] Costa, Therezinha Souza, Lecture Notes of Numerical Methods for Differential Equations, In Portuguese.

[3] Gear, C. W., "Numerical Initial Value Problems in Ordinary Differential Equations", Prentice-Hall (1971).

[4] Henrici. P, Discrete Variable Methods in Ordinary Differential Equations, John Wiley & Sons (1962).

[5] Lambert, J. D., Computional Methods in Ordinary Differential Equations, John Wiley & Sons (1972).

[6] Maurer, Stephen B., Ralston, Anthony. Discrete Algorithmic Mathematics, 3^{rd} Edition, A K Peters (2004).